An Analysis of Applying Fractal Dimension to Stream Networks

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Abstract

Fractal dimensions have been frequently applied to generalize the complexity and drainage of particular stream networks. Certain techniques to calculate dimension have become particularly popular, specifically the Divider method, functional-box counting, and as a function between bifurcation and length ratios. The frequency and variability with which these techniques are applied creates discrepancies as each technique applies different presumptions and display different fractal dimension values, skewing conclusions drawn on stream complexity. In addition, stream networks are assumed to be fractal without further validation. Analysis was conducted comparing the fractal dimensions at different sub-networks within a drainage basin to contrast the differing values between calculation techniques as well as to verify the fractality of the basin. Results indicated that the basin was fractal, but fractal dimensions between calculation techniques vary wildly and imply different relative complexities among sub-networks.

1. Introduction

As investigations in basin scale hydrological and geomorphic processes incorporate more themes in network theory, it becomes increasingly important to have a quantitative framework for analyzing network structure. Of the efforts to construct such a framework, *Horton [1932, 1945]* and *Strahler [1952]* were influential in pioneering a downstream system for ordering drainage networks. The Horton-Strahler ordering system has been useful in understanding stream networks as tree or dendritic networks *[Horsfield, 1980; Macdonald, 1983]*, and Horton used this ordering scheme to specify approximately constant bifurcation and length ratios for river channels.

In addition to stream ordering, fractals have also become important as a quantitative tool for assessing these tree networks. *Mandelbrot [1967, 1977, 1983]* first introduced fractals as a mathematical framework to treat complex and irregular geometries with similar patterns at varying scales. Thus, fractals have two main properties: self-similarity and heavy tails. Self-similarity describes the multiplicative invariance of an object's structure at multiple scales, while heavy tails illustrate that self-similarity undergoes a power-law. Fractals have played a major role in exploring influences at different scales by tree network structure on channel morphology and are frequently employed to quantify the complexity of river networks.

Though originally intended to understand dynamical systems with self-similarity in a mathematical context, fractals were soon reapplied to natural phenomena, such as organism sizing [*Buzsáki, 2013*] or landform development [*Prusinkiewicz, 1993*], as a useful basis to understand recurring patterns. It is important to note that mathematical fractals are strictly self-similar such that any subset of the whole fractal contains its exact geometry at a lower scale. This is to be contrasted with statistical fractals that exhibit self-affinity, where scaled invariances occur over a probability distribution. In understanding both mathematical and statistical fractals, fractal dimensions are used as a metric to describe the degree to which a pattern is scaled, such as the length of a coastline *Mandelbrot [1967]*. Though the fractal dimension perfectly describes the degree of pattern scale in mathematical fractals, it serves as a benchmark for the mathematical shape in natural patterns. Thus, non-integer dimensions have been very useful for characterizing the complexity of naturally formed boundaries and paths.

For example, works by La Barbera and Rosso *[1987]* as well as Rodriguez-Iturbe *[1988]* popularized the use of fractal dimensions to describe the levels of complexity among stream networks. With this popularity, multiple methods for calculating fractal dimensions have been developed for analyzing river networks, particularly the Divider (Richardson) method, functional box-counting, and the relation between bifurcation ratio and stream length ratio. To maximize the utility of analyzing fractal dimensions for understanding river networks, analysis is needed to identify the conceptual and numerical biases in different methods for characterizing fractals of river networks. In addition, the assignment of fractal dimensions across entire watersheds *[La Barbera and Rosso, 1989; Tarboton, 1988]* highlights the need for calculating dimensions of sub-networks within the same drainage area. An analysis at the sub-network level would both validate the presumption that fractal networks are inherently fractal as well as describe the probability distribution of a statistical fractal, depicting any skew or over-representation due to particular sub-networks.

In this paper, the application of various fractal dimension calculations on a Western Cascades stream network is shown to qualify its self-affinity. Variability is also tested among fractal dimension calculation techniques. Factors contributing to this variability and assumptions present in qualifying the fractality of a stream are described.

2. Methodology

2.1 Structure of Stream Networks

The Horton-Strahler ordering scheme is a downstream-moving ordering system used to classify channels in a stream network based on branching. Source channels are defined as first order streams where the convergence of two first order channels results in a second order channel. This pattern is repeated such that the joining of two channels of order w form a stream of order $w+1$, and at the joining of streams with different order, the downstream channel segment retains the higher order of the two streams. It is important to note that the Horton-Strahler ordering scheme presumes that stream network channels downstream of confluences are formed by the joining of exclusively two channels.

In conjunction with this ordering system, *Horton [1945]* introduced Horton's law of stream numbers, expressed by the bifurcation ratio, as well as Horton's law of stream lengths, expressed by the length ratio. Denoting $N(u\text{-}1)$ and $N(u)$ as the number of stream segments of orders u and $u\text{-}1$, respectively, the bifurcation ratio

$$R_b = \frac{N(u-1)}{N(u)} \tag{2}$$

describes a constant rate of bifurcation within a particular catchment. Horton's law of stream numbers also identifies the geometric relationship between a number of stream segments $N(i)$ given order i and highest order Ω, as described by

$$N(i) = Rb^{\Omega-i} \tag{2}$$

Similarly, the length ratio

$$R_l = \frac{L(u+1)}{L(u)} \tag{2}$$

also describes a constant scaling of stream length across a catchment and is defined by the geometric relationship

$$L(i) = L_1 Rl^{i-1} \tag{2}$$

where $L(i)$ is the mean length of streams with order i and L_1 is the mean length of first order streams.

2.2 Fractal Dimension of Networks

The self-similarity and self-affinity expressed in fractal objects are frequently defined by the fractal (Hausdorff) dimension, which describes the scaling property of a fractal attribute. Thus, the fractal dimension serves as a useful benchmark for quantifying and classifying the complexity of a fractal structure, and dimensions for river networks are frequently calculated to summarize the network's drainage.

Previous work deriving river network fractal dimensions have popularized three primary calculation techniques: the divider (Richardson) method, functional-box counting, and the relation between bifurcation ratio and stream length ratio.

2.2.1 Divider (Richardson) Method

Mandelbrot [*1983*] first estimates fractal dimensions in its reference to the length of Britain's coastline as presented by *Richardson* [1961]. To do so, the length of a shape is measured using a ruler of length r, such that the total measured length $L=Nr$ given N divider positive integer steps. Thus, the exact length is derived as $r\rightarrow\infty$ and the length converges to

$$L = \lim_{r\to 0} L(r) = \lim_{r\to 0} Nr \tag{1}$$

$$N \cong Lr^{-1} \tag{2}$$

However, *Richardson [1961]* found that the limit did not converge, so

$$\lim_{r \to 0} N(r) r^D = constant, D > 1 \tag{3}$$

implying that

$$N \sim r^{-D} \tag{4}$$

$$L \sim R^{1-D} \tag{5}$$

was derived where critical exponent *D* is the fractal dimension. When applied to stream networks, the fractal dimension is frequently assumed to be the Hausdorff dimension, defined as a limit covering a set by suitable subsets of decreasing diameter [*Moglen, 2002]*. Hence, the fractal dimension can be rewritten as

$$D = \lim_{r \to 0} \frac{\log N(r)}{\log(1/r)} \tag{6}$$

Tarboton [1988] measured the length of each Strahler stream, as defined according to Horton-Strahler ordering convention Strahler's [1952] network-ordering convention, and this is the procedure followed here. To account for the leftover end of streams during measurement, the distance is counted as another divider step if the length is greater than or equal to *r/2*.

2.2.2 Functional-Box Counting

To derive the function box-counting method of calculating fractal dimensions, a set of points is embedded in a *d*-dimensional space, where *d = 2* for river networks. The space is covered with a mesh if *d*-dimensional cubes with side length *r*. Thus, we have the relationship

$$n(r) \sim r^{-D} \tag{7}$$

where *n(r)* denotes the number of cubes containing the element to be considered. The fractal dimension is given by the slope of a log-log plot based on *Hentschel and Procaccia [1983]* given by the following

$$D = -\lim_{r \to 0} \lim_{m \to \infty} \frac{\log n(r)}{\log(r)} \tag{8}$$

2.2.3 Bifurcation Ratio vs Stream Length Ratio

Fractal dimension was derived as a function of the bifurcation law and the stream length ratio by *La Barbera and Rosso [1987]* using Horton's law of stream numbers as

$$D = \max(\frac{\log Rb}{\log \mathrm{Rl}}, 1) \tag{9}$$

where R_b is the bifurcation ratio and R_l is the length ratio. Thus, D is a single integer value as it compares two ratios already describing scaling.

2.3 Study Network

All data collected was from the H. J. Andrews Experimental Forest, a dense old-growth forest situated in the Western Cascade Range. The forest occupies over 6400-ha with a steep landscape at elevation ranging from 410 to 1630 meters and multiple peaks. The site covers the entire drainage basin of Lookout Creek, which feeds into the Blue River tributary of the McKenzie River system.

Fig 1. ArcGIS map of H.J. Andrews from 2008. The whole network is shown, with fourth order sub-networks colored green and red and third order sub-networks outlined.

In applying Strahler's ordering scheme for H.J. Andrews, seven sub-networks were created from the confluences merging fourth to fifth order streams and third to fourth order streams (Figure 1). The sub-networks are labeled "Whole" (fifth order), "McRae Creek" (fourth order), "Upper Lookout and Mack Creek" (fourth order), "North/South McRae" (third order), and "North/South Upper Lookout" (third order).

3. Results

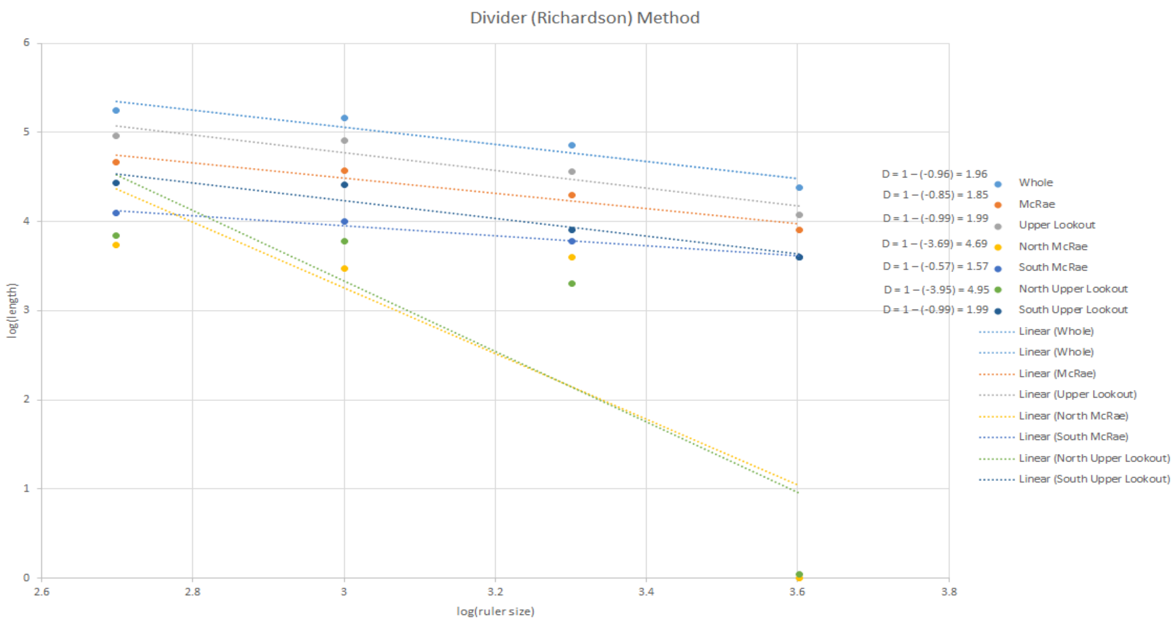
Table 1. An example comparison of Divider method scales and measured lengths for one of the seven sub-networks.

Stream	Ruler Length (m)	Counts	Total Length (m)	log(ruler)	log(length)
Whole	4000	6	24000	3.60205999	4.38021124
Whole	2000	36	72000	3.30103	4.8573325
Whole	1000	145	145000	3	5.161368
Whole	500	351	175500	2.69897	5.24427712

Using the divider method, four ruler lengths were chosen and scaled by ½, where the largest length was chosen to provide less than ten steps. Total measured lengths of each stream network are reported at different scales of rulers (Table 1).

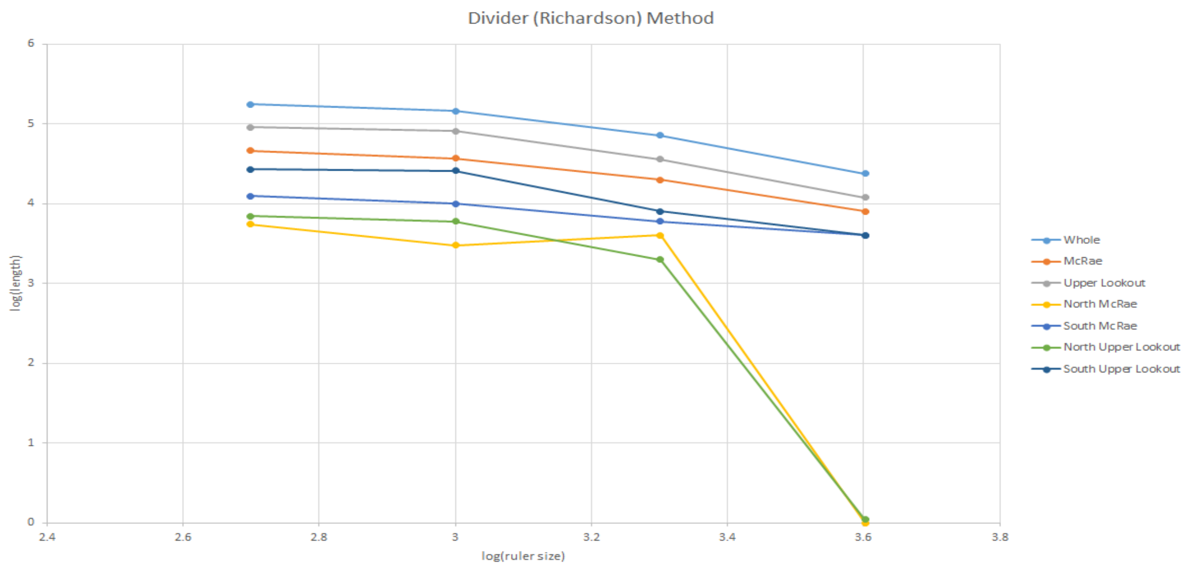
Graphing the scaling relation between the ruler length and measured stream length revealed that fractal dimensions to be between 1.57 and 4.95, where dimension D is represented as a function of the fitted slope (Figure 2). No relation was observed between the network order and dimension, but a sharp change in dimension was observed for North McRae and North Upper Lookout, both third order streams.

Fig. 2. Fractal dimension as a function of ruler length scales and the total measured stream length for all seven sub-networks. The dimension is provided as a function of slope.



To observe the distribution of slopes between ruler scales, the data were directly plotted, displaying steady fractal dimension relations at 500m and 1000m rulers followed by a sharp shift at between 2000m and 4000m rulers (Figure 3).

Fig. 3. Graphing the sharp change. The change in slope implies two separate dimensions that delineate sinuosity or branching.



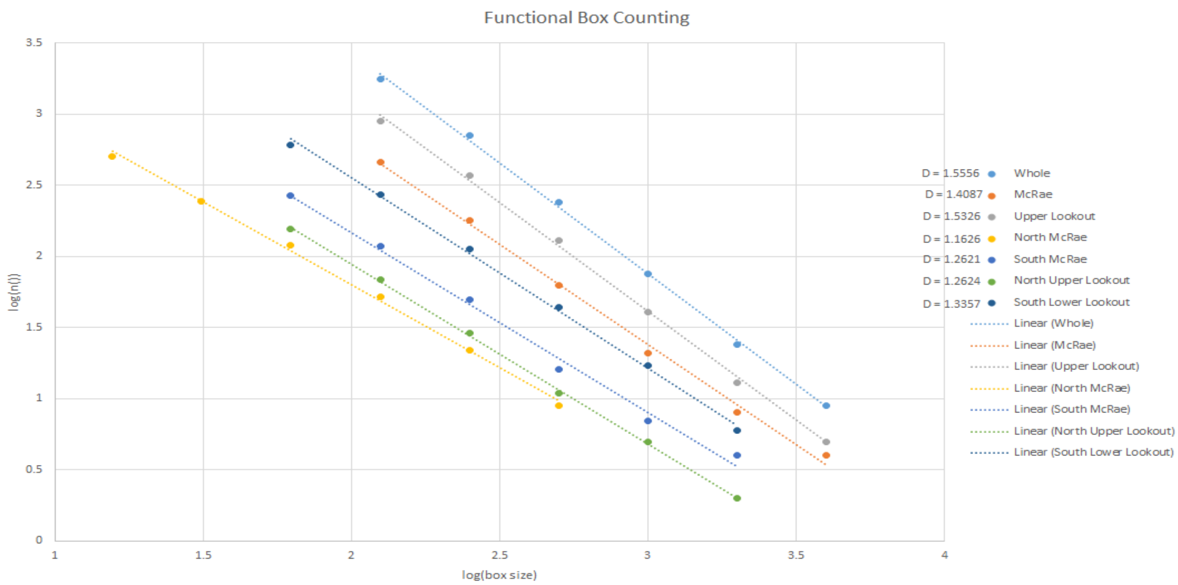
Functional box-counting was conducted similarly to the divider method. Six box sizings were used and scaled by ½, where the largest box size was chosen to provide a box count less than 10 (Table 2).

Table 2. An example comparison of box-counting scales and measured lengths for one of the seven sub-networks within H.J. Andrews.

Stream	Technique	Largest Order	Box Size (m)	Box Size Log	log(n)	n()
Whole	BoxCount	5	4000	3.602059991	0.954243	9
Whole	BoxCount	5	2000	3.301029996	1.380211	24
Whole	BoxCount	5	1000	3	1.875061	75
Whole	BoxCount	5	500	2.698970004	2.383815	242
Whole	BoxCount	5	250	2.397940009	2.85309	713
Whole	BoxCount	5	125	2.096910013	3.246745	1765

Fractal dimension calculations for each sub-network demonstrated a consistent decrease in dimension with lower ordered networks with dimensions ranging between 1.16 and 1.55 (Figure 4). Fractal dimension was largely uniform between networks, with the range resembling river network data from existing literature [*Tarboton, 1988*], and values derived from functional box-counting maintain lower values to that of the divider method.

Fig. 4. Fractal dimension as a function of box size scales and the number of boxes containing stream segments for all seven sub-networks. The dimension is provided as the slope.



Fractal dimensions derived as a function of bifurcation ratios and stream length ratios were calculated using data from arcMaps provided by H.J. Andrews with stream order classifications (Table 3).

Table 3. An example comparison of bifurcation ratios and stream length ratios, including the associated fractal dimension, for the whole H.J. Andrews river sub-network.

Order	Count	Average length	log(count)	log(length)	log(Rb)	log(Rl)	Dimension
5	1	6844.9044	0	3.835367	0.661063	0.346309	1.908883
4	2	5584.39055	0.30103	3.746976			
3	14	1292.144464	1.146128	3.111311			
2	71	694.8318521	1.851258	2.84188			
1	339	360.0579165	2.5302	2.556372			

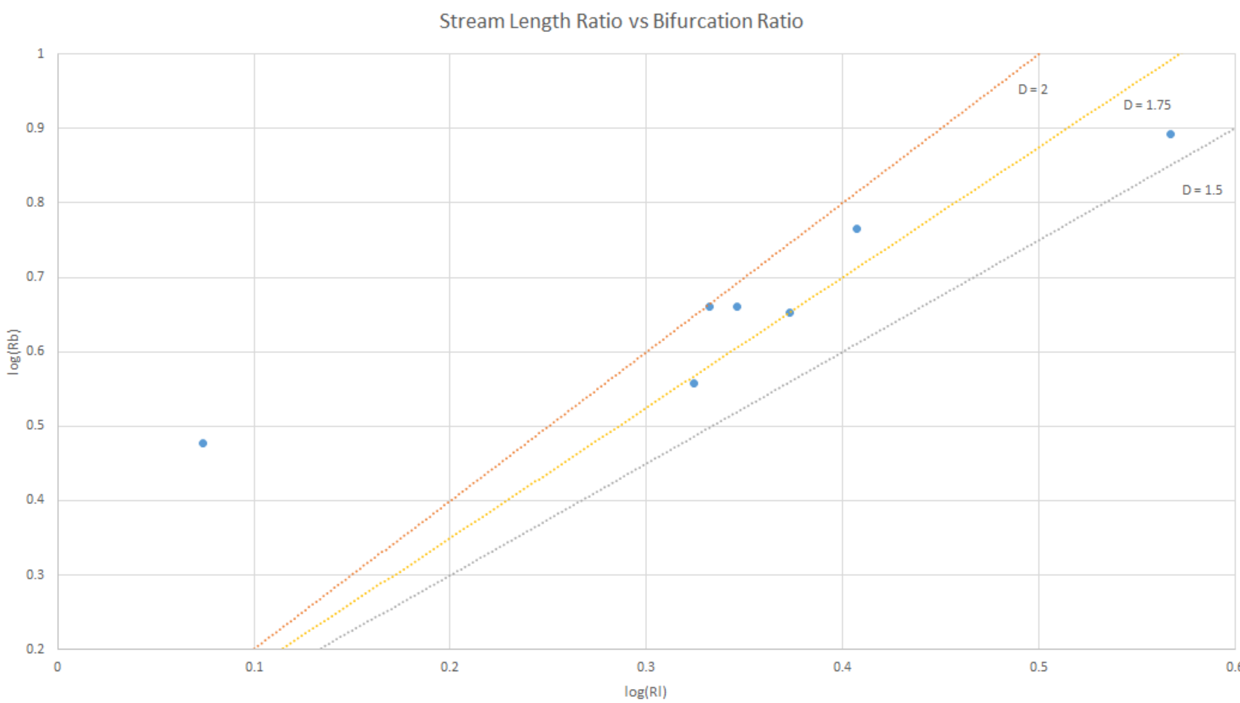
The dimensions between sub-networks held values from 1.57 to 6.43, where the outlier dimension of 6.43 belonged to the North McRae sub-network, which expressed similarly high dimension as calculated using the divider method (Table 4). It was noted that North Upper Lookout, which also expressed an outlying high dimensionality via divider method, did not express distinctly large dimensions when calculated by the bifurcation and length function.

Table 4. An example comparison of bifurcation ratios and stream length ratios, including the associated fractal dimension, for the whole H.J. Andrews river sub-network.

Stream	Dimension
Whole	1.91
McRae	1.75
Upper Lookout	1.88
North McRae	6.43
South McRae	1.99
North Upper Lookout	1.72
South Upper Lookout	1.57

Similarly to dimensions from the divider method, calculated fractal dimensions did not display a relation between sub-network order and fractal dimension, and values generally lied between 1.75 and 2, which match existing literature predicting river networks to have dimensions near 2 [*Tarboton, 1988*].

Fig. 4. Fractal dimension as a function of bifurcation ratios and stream length ratios for each of the sub-networks in H.J. Andrews. The comparison of box-counting scales and measured lengths, including the associated fractal dimension, for each of the seven sub-networks within H.J. Andrews.



4. Discussion

Overall, the H.J. Andrews stream network expressed statistical fractality. Excluding outliers, each calculation technique expressed the fractal dimensions of all sub-networks with <0.5 sized ranges. This result is similarly reflected in prior work by Tarboton [*1988*] with the sub-network fractal dimension range for each technique mirroring dimension value ranges comparing whole drainage basins. La Barbera and Rosso [1989] also provides theoretical support for the resulting stream network dimension values existing between 1 and 2. However, the calculation techniques still provide different values from each other, implying different presumptions of how space-filling the sub-network may be.

The relationship between different fractal dimensions was not equivalent between techniques because functional box-counting showed a distinct decreasing trend among dimensions as the order of the sampled sub-network decreased, but this pattern was not exhibited by either the Divider method or the bifurcation and length function. Thus, conclusions drawn to categorize the branching scale or space-filling of different networks are highly dependent on the applied calculation method. In addition, scaling factors are selectively expressed by different functions. For example, sharp increases in dimension applied by the Divider method were similarly encountered in Tarboton [*1988*], which cited the dimension increase to be a function of the stream branching being more expressed than stream sinuosity. Tarboton further demonstrated that a piecewise function needed to be used to describe how different stream geometries and shapes produce multiple fractal dimensions.

In accounting for branching and sinuosity as separate variables in fractal dimensions, parallels can be found with the geomorphology of the sub-networks. The use of fractal dimensions as a parallel to drainage density may apply when observing the North and South Upper Lookout sub-networks. It was shown by the Divider method that North Upper Lookout expressed a sharply high fractal dimension, suggesting overrepresented branching, whereas South Upper Lookout expressed a steady and lower fractal dimension, suggesting more sinuosity. This coincides well with the historical development of these sub-network geomorphologies as North Upper Lookout development was heavily influenced by earthflow movements that force a high degree of bifurcation in the stream. This is contrasted by glacial influences on South Upper Lookout, which produced a more fern-like network structure that pronounce a single main river stem. However, it must be noted that given the multiple areas of disagreement among fractal dimension techniques, it is unreliable to relate a network's dimension to geomorphological patterns.

5. Conclusions

Quantitative approaches to understanding stream networks have been very useful in treating stream complexity rigorously. Applying fractal dimensions to stream networks produces broad insights on the drainage density of particular watersheds, but care must be taken in selecting a calculation technique to accommodate for appropriate presumptions on network structure while also accounting for heavy skew by individual sub-networks. This study implicates more interesting questions on the quantitative difference branching and sinuosity have on fractal dimension and also demands a more rigorous representation of the probabilistic distribution relating a statistical fractal.

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References

B. B. Mandelbrot, “How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension,” Science, New Series, Vol. 156, No. 3775, 1967, pp. 636-638.

Buzsáki G, Logothetis N, Singer W. “Scaling brain size, keeping time: evolutionary preservation of brain rhythms.” Neuron. 2013;80:751–64.

Fractal River Basins: Chance and Self-Organization by Ignacio Rodríguez-Iturbe and Andrea Rinaldo (Eds.) Cambridge University Press, Cambridge, UK, 547pp ISBN-0-521-47398-5 (hardback) Published 1997; ISBN 0-521-00405-5 (paperback) Published 2001. Hydrol. Process., 16: 3097–3098. doi:10.1002/hyp.5059

Horton, R. E. 1932. ‘Drainage-basin characteristics,” *EOS Trans. AGU*, **13,** 350-361.

I. Rodriguez and A. Rinaldo, “Fractal River Basins (Chance and Self-Organization),” Cambridge University Press, Cambridge, 1997.

La Barbera, P., and R. Rosso. 1990. “On fractal dimension of stream networks,” Reply to Tarboton et al., *Water. Resour. Res.*, **26**(9), 2245-2248.

Lovejoy, S., D. Schertzer, and A. A. Tsonis. 1987. Functional box counting and multiple elliptical dimensions in rain, *Science*, **235,** 1036-1038.

Prusinkiewicz, P. and Hammel, M. 1993. “Fractal model of mountains with rivers.” In Proceeding of the Canadian Conference on Graphics Interface. 174--180.

Strahler, A. N. 1952. “Hypsometric (area altitude) analysis of erosional topography”, *Geol. Soc. Am. Bull.*, **63**, 1117-1142.

Spies, T. 2016. LiDAR data (August 2008) for the Andrews Experimental Forest and Willamette National Forest study areas. Long-Term Ecological Research. Forest Science Data Bank, Corvallis, OR. [Database].

Tarboton, D. E., R. L. Bras, and I. Rodríguez-Iturbe, "The Fractal Nature of River Networks," Water Resources Research, 24(8):1317-1322, 1988.

Z. Khanbabaei, A. Karam, and G. Rostamizad, "Studying relationships between the fractal dimension of the drainage basins and some of their geomorphological characteristics", *Int. J. Geosci.*, vol. 4, pp. 636-642, 2013.